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Diffusive processes in a stochastic magnetic field

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The statistical representation of a fluctuating (stochastic) magnetic field configuration is studied in detail. The Eulerian correlation functions of the magnetic field are determined, taking into account all geometrical constraints: these objects form a nondiagonal matrix. The Lagrangian correlations, within the reasonable Corrsin approximation, are reduced to a single scalar function, determined by an integral equation. The mean square perpendicular deviation of a geometrical point moving along a perturbed field line is determined by a nonlinear second-order differential equation. The separation of neighboring field lines in a stochastic magnetic field is studied. We find exponentiation lengths of both signs describing, in particular, a decay (on the average) of any initial anisotropy. The vanishing sum of these exponentiation lengths ensures the existence of an invariant which was overlooked in previous works. Next, the separation of a particle's trajectory from the magnetic field line to which it was initially attached is studied by a similar method. Here too an initial phase of exponential separation appears. Assuming the existence of a final diffusive phase, anomalous diffusion coefficients are found for both weakly and strongly collisional limits. The latter is identical to the well known Rechester-Rosenbluth coefficient, which is obtained here by a more quantitative (though not entirely deductive) treatment than in earlier works.

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I. INTRODUCTION

The problem of transport of particles and energy in a magnetically confined plasma, in a region where the magnetic field is completely stochastic, is of major importance in the problem of controlled fusion [1], as well as in other fields of plasma physics. This matter has been the object of many previous studies (see, especially, Refs. [2–27]), but has not yet yielded a final answer, because it is a quite complex nonlinear problem. We intend to devote a series of papers to an alternative approach to this question.

In order to concentrate on the statistical aspects of the problem, we consider an extremely simple geometry. The main magnetic field is supposed to be straight, homogeneous, and stationary. It therefore fixes a privileged direction, which is taken as the z axis of a Cartesian reference frame. It might be objected that this shearless slab geometry is not very realistic when applied to a fusion device such as the tokamak; previous authors have studied the problem in sheared slab [3,4,8,11,12,14,15,21], cylindrical [2,5,9,12,22,26], or toroidal [20] geometries. Although the presence of shear is important for the emergence of chaos mechanism through tearing modes, it appears that in a completely chaotic situation its role is no longer essential. It is noteworthy that the classical formulas for the diffusion coefficient, as derived, e.g., by Rechester and Rosenbluth [5] by using a cylindrical geometry, do not depend on the inhomogeneity or the shear of the unperturbed magnetic field. We thus decided to radically simplify the geometrical aspects of the problem (as in Refs. [7,10,17–19,25,27]) and to consider the influence of the geometry in forthcoming work.

The anomalous transport is due to a superposition of three stochastic processes. In the first place we must consider the ubiquitous collisions producing random velocities in the parallel and in the perpendicular directions, which, if they were alone, would determine the well known classical transport phenomena [28]. The magnetic field, on the other hand, is supposed to contain a fluctuating component, perpendicular to the main field, due to internal instabilities or to irregularities in the external coils ("braided field"). The total magnetic field in such a stochastic layer must be described statistically. The magnetic field lines are no longer deterministic in this case. Their behavior must rather be described as a spatial diffusion in the x-y plane. The motion of the particles along this perturbed, diffusing field combined with the collisions that may decorrelate them from the magnetic field eventually yields enhanced, anomalous transport.

The first step in any theory of magnetic fluctuations is the statistical definition of the magnetic field, which is considered as a random variable. In all papers (except Ref. [8]) the magnetic field is assumed to be statistically prescribed *a priori* as a Gaussian colored process; the retroaction of the plasma on the field is not considered.

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Such a simplified picture can be justified by the great difficulty of a fully self-consistent theory. A complete specification of a Gaussian process requires a definition of the correlation functions of the magnetic field evaluated at two fixed points in space, i.e., the Eulerian correlation functions. In most works [5–7,9–11,13,14,19–26] the form of the fluctuation spectrum is not explicitly specified; the fluctuations are characterized solely by their intensity and the characteristic length scales in the directions parallel and perpendicular to the main field. To the best of our knowledge, an explicit definition is given only in Refs. [12,17,18]; it will be shown in Sec. II that the definitions of Refs. [14,15,17,18] are inconsistent with the existing physical constraints.

The form and properties of the various moments of a random vector field (including the condition of zero divergence) are discussed in most texts on fluid turbulence theory. The general form of the Eulerian correlation tensor for the present problem has been derived in detail by Coronado, Vitela, and Akcasu [19]. Specifying in their results the form of the autocorrelation functions, we derive in Sec. II explicit formulas for the Eulerian correlation functions of the magnetic field and of the gradient of this field.

In the study of the diffusion process, the central quantity is the mean square deviation (MSD) of the position of a point (or a particle) averaged over the ensemble of realizations of the random variable. This quantity, in turn, is determined by the Lagrangian correlation function of the fluctuating magnetic field. In this correlation function the field is no longer evaluated at two fixed points, but rather at two successive positions along an orbit in a given realization, averaged over these realizations. The Lagrangian correlations are quite complex objects, as is well known also in fluid turbulence theory [29,30], because they require the solution of the equations of motion, a problem that is usually impossible. Their evaluation requires reasonable approximations, such as the well known Corrsin approximation [30,31]. The Lagrangian correlations of the magnetic fluctuations are discussed in Sec. III. It is shown that, within the Corrsin framework, all these correlations are determined by a single scalar function, which obeys an elegant integral equation. The latter can be solved exactly in a certain limit and numerically in general.

Section IV is devoted to the study of the spatial diffusion of magnetic field lines. A second-order nonlinear differential equation is derived for the MSD. From its solution, the diffusion coefficient of the magnetic field lines can be derived. The equation can be solved analytically in two extreme cases (quasilinear and percolation limits). Although a well known result is recovered in the latter limit, it is shown that this result is illusory because the differential equation is not valid in the region of large perturbation; the correct behavior must be obtained by other methods.

In Sec. V we study the relative motion of the field lines. Instead of considering the displacement of a real field line with respect to an (abstract) average line, we consider here initially ($\zeta=0$) two neighboring magnetic field lines and study their relative distance along ζ. (Here ζ represents the z coordinate running along the straight unperturbed magnetic field and playing the role of time in this purely geometrical problem.) The result is qualitatively well known: the relative separation grows exponentially with ζ (in at least one direction), a signature of a chaotic system. The characteristic length scales associated with this process are the (spatial) Lyapounov exponents (or exponentiation lengths). Though there have been three previous detailed studies of this problem [11,12,14], the results of Refs. [11,14] were not correct because they did not take properly into account the geometrical constraints in calculating the Eulerian correlation functions of the magnetic field lines.

In Sec. VI we introduce charged particles into this fluctuating magnetic field configuration. These particles are constituents of a plasma; in other words, they are not considered as isolated. Rather, they move under the combined action of the stochastic magnetic field and of their mutual collisions. The effect of the latter is modeled here as a random component of the velocity and the global evolution is described by a stochastic equation of motion. The philosophy of this model has been discussed in great detail in our previous paper [27], where we used the terminology "V-Langevin equation" for the latter equation and studied a particular (subdiffusive) case of this problem.

Here we consider the general case, in which $\boldsymbol{b}=\boldsymbol{b}(x,y,z)$. In a pioneering paper, Rechester and Rosenbluth [5] showed that in this case the particles behave diffusively, i.e., their mean square displacement is asymptotically proportional to time. The treatment in Ref. [5] was, however, semiphenomenological. None of the subsequent authors working on this problem [11,14,18,23] succeeded in obtaining a rigorous derivation of the Rechester-Rosenbluth (RR) diffusion coefficient from first principles. An exception is the paper of Laval [21], who gave an analytic treatment leading (in particular in his Eq. 42) to a result very similar to the RR diffusion coefficient. He used, however, an idealized model for the magnetic field whose relation to a real configuration is very hard to evaluate. We do not claim to present a rigorous proof of the RR result; we believe, however, that the point of view adopted here sheds light on this problem. Instead of considering the deviation of the particle trajectory from its average, we study a problem analogous to the one of Sec. V. In the latter we computed the relative separation of two field lines; here we calculate, as a function of time, the relative distance between a physical particle trajectory and the field line on which it started at time zero. This way of treating the problem clarifies the mechanism of decorrelation of the particles from the field lines. It appears, in particular, that a perpendicular collisional diffusion coefficient, although very small compared to the parallel one in a very strong magnetic field, plays the role of a necessary "seed" in order to produce the decorrelation that finally leads to a strongly enhanced anomalous diffusion coefficient. This conclusion was also reached in a rather different way by Rechester and Rosenbluth [5], Isichenko [14], and Laval [21]. It may be added that an alternative mechanism of decorrelation is provided by the perpendicular drift motions, which were neglected here. Coronado, Vitela, and Akcasu [19] formulated this problem, but did not derive explicit expressions for the contribution of this effect to the anomalous diffusion coefficient. We extended our methods to cover this problem and will present the results in a forthcoming paper.

II. KINEMATICS OF THE MAGNETIC FLUCTUATIONS

The magnetic field considered in the present paper consists of a strong constant field $\boldsymbol{B}_0$ directed (conventionally) along the z axis of a Cartesian reference frame $[\boldsymbol{x}\equiv(x,y,z),\ \boldsymbol{x}_\perp\equiv(x,y)]$ and a small perturbating field perpendicular to the former:

$$\boldsymbol{B}(\boldsymbol{x})=B_0[\boldsymbol{e}_z+b_x(\boldsymbol{x})\boldsymbol{e}_x+b_y(\mathbf{x})\boldsymbol{e}_y]\ . \qquad (1)$$

(In the present paper the definitions of "parallel" and "perpendicular" directions refer to the unperturbed field direction $\boldsymbol{e}_z$.) In contrast with most papers quoted in Sec. I, our unperturbed field has no shear. The present shearless slab model has the advantage of simplicity and does not lack any essential ingredient for understanding a possible (though not the only possible) mechanism of anomalous transport. As explained in Ref. [19], this idealization is not too bad whenever the correlation lengths are much smaller than the characteristic plasma dimensions: the phenomena can then be described locally. (A formulation valid in a very general geometry was given in Ref. [12].) Clearly, the magnetic field must obey the constraint of zero divergence: $\nabla\cdot\boldsymbol{B}(\boldsymbol{x})=0$. In order to automatically satisfy this constraint, we represent the perturbing field in terms of a vector potential $\boldsymbol{a}(\boldsymbol{x})$ having only a z component:

$$\boldsymbol{a}(\boldsymbol{x})=\psi(\boldsymbol{x})\boldsymbol{e}_z,\quad \boldsymbol{b}(\boldsymbol{x})=\nabla\times\boldsymbol{a}(\boldsymbol{x})\ . \qquad (2)$$

For any field $C(\boldsymbol{x})$ we define its Fourier transform $C(\boldsymbol{k})$ by

$$C(\boldsymbol{x})=\int d\boldsymbol{k}\ e^{i\boldsymbol{k}\cdot\boldsymbol{x}}C(\boldsymbol{k})\ . \qquad (3)$$

We now state that the perturbing magnetic field $\boldsymbol{b}(\boldsymbol{x})$ is a random quantity that has to be defined statistically. In order to cope with the zero-divergence constraint, we introduce the primary definition at the level of the potential $\psi(\boldsymbol{x})$: this quantity is assumed to be a Gaussian random function, which is supposed to be spatially homogeneous and gyrotropic [i.e., isotropic in the plane perpendicular to the reference magnetic field]. In this case, the Eulerian potential autocorrelation function has the form

$$\mathcal{A}(\boldsymbol{r})=\langle\psi(\boldsymbol{x}+\boldsymbol{r})\psi(\boldsymbol{x})\rangle=\langle\psi(\boldsymbol{r})\psi(0)\rangle=\mathcal{A}(r_\perp,r_z)\ , \qquad (4)$$

where the scalar function $\mathcal{A}(\boldsymbol{r})$ depends only on the length of the perpendicular component of the relative distance $r_\perp=\sqrt{r_x^2+r_y^2}$ and on the z component r_z. We stress the attribute "Eulerian," meaning that the correlation is evaluated at two fixed points $\boldsymbol{x}+\boldsymbol{r},\boldsymbol{x}$ in physical space. In the Fourier representation, we introduce the potential spectral density or, briefly, the potential spectrum:

$$\langle\psi(\boldsymbol{k})\psi(\boldsymbol{k}')\rangle=\mathcal{A}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}')\equiv\mathcal{A}(k_\perp,k_\parallel)\delta(\boldsymbol{k}+\boldsymbol{k}')\ , \qquad (5)$$

where $k_\perp=\sqrt{k_x^2+k_y^2}$, $k_\parallel=k_z$.

We now consider the four possible correlation functions of the magnetic field components in the Fourier representation $b_m(\boldsymbol{k})$, $m=x,y$, which form a 2×2 matrix:

$$\langle b_m(\boldsymbol{k})b_n(\boldsymbol{k}')\rangle=\mathcal{B}_{mn}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}')\ , \qquad (6)$$

with

$$\mathcal{B}_{mn}(\boldsymbol{k})=(k_\perp^2\delta_{mn}-k_mk_n)\mathcal{A}(\boldsymbol{k})\ . \qquad (7)$$

From these expressions, we can easily derive the Eulerian correlations

$$\mathcal{B}_{mn}(\boldsymbol{r})=\langle b_m(\boldsymbol{r})b_n(0)\rangle=\int d\boldsymbol{k}\ e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathcal{B}_{mn}(\boldsymbol{k})\ . \qquad (8)$$

These quantities can be easily expressed explicitly in terms of the potential spectrum. We do not give here the detailed derivation of this result, which was discussed in detail in Ref. [19]:

$$\mathcal{B}_{mn}(\boldsymbol{r})=\mathcal{I}_1(r_\perp)\delta_{mn}+\mathcal{I}_2(r_\perp)(r_\perp^2\delta_{mn}-r_mr_n)\ . \qquad (9)$$

Here $\mathcal{I}_1,\mathcal{I}_2$ are functions of $r_\perp$ that can be calculated explicitly in terms of the potential spectrum $\mathcal{A}(\boldsymbol{k})$. Equation (9) represents the general form of the homogeneous, gyrotropic Eulerian correlation function for a solenoidal (divergence-free) fluctuating, two-dimensional vector field. No assumption has been made about any specific form of the potential spectrum $\mathcal{A}(\boldsymbol{k})$. It should be noted, in particular, that the correlation tensor is, in general, nondiagonal: the integral $\mathcal{I}_2$ is nonzero in most of the relevant situations (see below). This is in contradistinction to the assumptions often found in the literature [14–18].

We now make the model more specific by introducing an assumption about the form of the potential spectrum (a similar assumption can be found explicitly in Ref. [18]):

$$\mathcal{A}(\boldsymbol{k})=(2\pi)^{-3/2}\lambda_\parallel\lambda_\perp^4\beta^2\exp(-\tfrac{1}{2}\lambda_\parallel^2k_\parallel^2-\tfrac{1}{2}\lambda_\perp^2k_\perp^2)\ . \qquad (10)$$

The corresponding correlation function in physical space is

$$\mathcal{A}(\boldsymbol{r})=\beta^2\lambda_\perp^2\exp\left[-\frac{r_z^2}{2\lambda_\parallel^2}-\frac{r_\perp^2}{2\lambda_\perp^2}\right]\ . \qquad (11)$$

The potential spectrum thus depends on two characteristic lengths: the parallel correlation length $\lambda_\parallel$ and the perpendicular correlation length $\lambda_\perp$; the dimensionless parameter β is a measure of the intensity of the fluctuations. With assumption (10) the functions $\mathcal{I}_p$ of Eq. (9) can be calculated rather easily, with the following result for the magnetic field correlation tensor:

$$\mathcal{B}_{mn}(\boldsymbol{r})=\beta^2\left\{\delta_{mn}-\frac{r_\perp^2\delta_{mn}-r_mr_n}{\lambda_\perp^2}\right\}\times\exp\left[-\frac{r_z^2}{2\lambda_\parallel^2}-\frac{r_\perp^2}{2\lambda_\perp^2}\right],\quad m,n=x,y\ . \qquad (12)$$

Explicit expressions can also be obtained for the Eulerian correlations of gradients of the magnetic field (these quantities will be needed hereafter). Introducing the abbreviation $b_{m,\alpha}(\mathbf{x})=\nabla_\alpha b_m(\mathbf{x})$, $m,\alpha=x,y$, we define the following correlation tensors (of fourth rank):

$$\mathcal{B}_{mn}^{\alpha\beta}(\mathbf{r})=\langle b_{m,\alpha}(\mathbf{x}+\mathbf{r})b_{n,\beta}(\mathbf{x})\rangle \ . \tag{13}$$

Using the Fourier transforms of the fluctuating field we find

$$\mathcal{B}_{mn}^{\alpha\beta}(\mathbf{r})=\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}k_\alpha k_\beta \mathcal{B}_{mn}(\mathbf{k}) \ . \tag{14}$$

The result is easily obtained from Eqs. (7) and (10). We do not list here the explicit formulas, but mention that this tensor turns out to possess five independent components $\mathcal{B}_{xx}^{xx}(\mathbf{r})$, $\mathcal{B}_{xx}^{yy}(\mathbf{r})$, $\mathcal{B}_{yy}^{xx}(\mathbf{r})$, $\mathcal{B}_{xx}^{xy}(\mathbf{r})$, and $\mathcal{B}_{yy}^{xy}(\mathbf{r})$ (i.e., these components have different functional dependence on $\mathbf{r}$); every other component of the tensor equals $\mp$ one of these, due to the symmetries $\mathcal{B}_{mn}^{\alpha\beta}(\mathbf{r})=\mathcal{B}_{mn}^{\beta\alpha}(\mathbf{r})=\mathcal{B}_{nm}^{\alpha\beta}(\mathbf{r})=\mathcal{B}_{nm}^{\beta\alpha}(\mathbf{r})$.

We now compare these results with those appearing in the literature. The most explicit presentation of this matter is found in Ref. [14]; Isichenko (IS) assumes the following form:

$$\mathcal{B}_{mn}^{\alpha\beta}(r_z)=\delta_{nm}\delta_{\alpha\beta}f_{\alpha m}(r_z) \quad \text{(IS)} \ . \tag{15}$$

When the b_z component is zero (as in our case) his assumption is [Eq. (11) of Ref. [14], translated into our notation]

$$\int_{-\infty}^{\infty} dz\, f_{\alpha m}(z)\approx 2\beta^2\frac{\lambda_\parallel}{\lambda_\perp^2} \quad \forall\alpha,\ m=x,y \quad \text{(IS)} \ . \tag{16}$$

This result is obtained by a purely dimensional argument, from the mere existence of two characteristic scale lengths. The same assumption appears in Ref. [11]. The main difference between our result (14) and Eqs. (15) and (16) is that the latter correlation function $\mathcal{B}_{mn}^{\alpha\beta}$ depends solely on r_z instead of being a function of both r_z and $\mathbf{r}_\perp$. In order to obtain Isichenko's $\mathbf{r}_\perp$-independent result, we must admit (as stated by him) that the magnetic field correlation is "insensitive to the transverse correlation length" $\lambda_\perp$: this is the case when $\lambda_\perp\to\infty$. Thus the implicit approximation admitted by Isichenko [14] and by the other authors quoted is $\lambda_\parallel/\lambda_\perp \ll 1$. In this approximation the magnetic fluctuation problem is enormously simplified; this case was treated in great detail in Ref. [27] (see also the discussions in the forthcoming sections). Many important features related to the specific nonlinearity of the problem are, however, wiped out in this approximation. Its discussion is therefore not very useful for a general, realistic situation.

If we admitted Isichenko's approximation $\lambda_\perp\to\infty$, we should neglect $r_\perp/\lambda_\perp$ in all correlation functions; the Eulerian magnetic field correlation (12) becomes

$$\mathcal{B}_{mn}(r_z)=\mathcal{B}(r_z)\delta_{mn}=\beta^2\exp\left[-\frac{r_z^2}{2\lambda_\parallel^2}\right]\delta_{mn} \ , \qquad \frac{\lambda_\parallel}{\lambda_\perp}\ll 1 \ . \tag{17}$$

In this limit (and only in this limit) the Eulerian field correlation tensor becomes diagonal. In order to calculate the correlation of the gradients, one must first calculate the exact expressions (14) and then take the limit $r_\perp/\lambda_\perp \ll 1$, with the result

$$\begin{aligned}\mathcal{B}_{xx}^{xx}(r_z)=\mathcal{B}_{yy}^{yy}(r_z)=\tfrac{1}{3}\mathcal{B}_{xx}^{yy}(r_z)&\\ =\tfrac{1}{3}\mathcal{B}_{yy}^{xx}(r_z)=-\mathcal{B}_{xy}^{xy}(r_z)&\\ =\frac{\beta^2}{\lambda_\perp^2}\exp\left[-\frac{r_z^2}{2\lambda_\parallel^2}\right]&\\ \text{for } \frac{\lambda_\parallel}{\lambda_\perp}\ll 1 \ .&\end{aligned} \tag{18}$$

Hence, even in this limiting case, the result is different from Isichenko's. In the first place, the functions $f_{\alpha m}(r_z)$ in Eq. (15) are different for different indices; more important, there exists a nondiagonal coefficient $\mathcal{B}_{xy}^{xy}(r_z)$. Indeed, $\langle b_{x,x}b_{x,x}\rangle=-\langle b_{x,x}b_{y,y}\rangle$ (because $\nabla\cdot\mathbf{b}=0$ and $b_z=0$); hence $\mathcal{B}_{xx}^{xx}\neq 0$ implies $\mathcal{B}_{xy}^{xy}\neq 0$. Note that nondiagonal components of this tensor were considered in Ref. [12]. We thus see that in all cases one must correctly take into account the constraint of zero divergence.

III. LAGRANGIAN CORRELATION FUNCTIONS

In Sec. II we discussed the Eulerian correlation functions of the magnetic fluctuations. In the evolution and transport problems that constitute our main objective, another type of correlation functions plays a major role. We introduce the latter at two levels.

We first consider a field line representing the (given) magnetic field as a purely geometric object. It is a curve in three-dimensional space, represented by the following equations [see Eq. (1)]: $dx/(B_0b_x)=dy/(B_0b_y)=dz/B_0$. We take z as an independent variable (playing the same role as the time in a dynamical problem) and denote it by the greek letter ζ in order to stress its distinct role; we then obtain

$$\begin{aligned}\frac{dx(\zeta)}{d\zeta}&=b_x(x(\zeta),y(\zeta),\zeta) \ ,\\ \frac{dy(\zeta)}{d\zeta}&=b_y(x(\zeta),y(\zeta),\zeta) \ .\end{aligned} \tag{19}$$

Similar equations valid for a general geometry of the unperturbed field were derived in Ref. [12].

When these equations are combined with Eqs. (2), one clearly sees that they have a Hamiltonian structure, with (x,y) playing the role of (q,p), and with a ζ-dependent (i.e., time-dependent) Hamiltonian, $\psi(x,y;\zeta)$. The system is therefore nonautonomous, hence, in general, nonintegrable [$1\frac{1}{2}$ degrees of freedom]. Two limiting situations are, however, solvable. When $\mathbf{b}(\zeta)$ only depends on ζ, the equations are trivially simple: this case has been thoroughly discussed in Ref. [27]. When $\mathbf{b}(x,y)$ is independent (explicitly) of ζ, we have an integrable, autonomous one-degree-of-freedom system.

Writing $\mathbf{x}_\perp(\zeta)\equiv(x(\zeta),y(\zeta))$, we integrate Eq. (19):

$$\delta\mathbf{x}_\perp(\zeta)=\int_0^\zeta d\zeta_1\mathbf{b}(\mathbf{x}_\perp(\zeta_1),\zeta_1) \ , \tag{20}$$

where we introduced the deviation $\delta\boldsymbol{x}_\perp(\zeta)$ defined as follows:

$$\delta\boldsymbol{x}_\perp(\zeta)=\boldsymbol{x}_\perp(\zeta)-\langle \boldsymbol{x}_\perp(\zeta)\rangle=\boldsymbol{x}_\perp(\zeta)-\boldsymbol{x}_\perp(0) \; . \tag{21}$$

From the solution (20) we deduce the MSD in the x direction, at "time" ζ:

$$\Gamma(\zeta)\equiv\langle \delta x^2(\zeta)\rangle=\int_0^\zeta d\zeta_1\int_0^\zeta d\zeta_2\mathcal{L}_{xx}(\zeta_1,\zeta_2)$$
$$=2\int_0^\zeta d\bar{\zeta}(\zeta-\bar{\zeta})\mathcal{L}_{xx}(\bar{\zeta}) \; . \tag{22}$$

The integrand is the (xx) component of the Lagrangian correlation tensor of the magnetic field fluctuations: it is an ensemble average of the product of fluctuating fields evaluated at the instantaneous positions $\boldsymbol{x}_\perp(\zeta)$ at two different "times" ζ_1,ζ_2 (rather than at fixed spatial positions $\boldsymbol{x}_\perp,z$ as in the Eulerian correlations):

$$\mathcal{L}_{mn}(\zeta_1+\zeta,\zeta_1)$$
$$=\langle b_m(\boldsymbol{x}_\perp(\zeta_1+\zeta),\zeta_1+\zeta)b_n(\boldsymbol{x}_\perp(\zeta_1),\zeta_1)\rangle$$
$$=\langle b_m(\boldsymbol{x}_\perp(\zeta),\zeta)b_n(\boldsymbol{x}_\perp(0),0)\rangle\equiv\mathcal{L}_{mn}(\zeta) \; . \tag{23}$$

The second equality follows from a theorem of Lumley [32] (see also Ref. [30]): when the Eulerian correlation function of the fluctuating field is homogeneous and stationary, the Lagrangian correlation is stationary (i.e., depends only on the difference of the two times).

Lagrangian correlations are much more complicated mathematical objects than Eulerian correlations: this is well known in the context of fluid turbulence theory (an excellent recent discussion can be found in Chap. 12 of Ref. [30]). The reason is easily understood: one needs the solution of the equations of motion (19) for its calculation.

It is customary in turbulence theory to use an approximation procedure due to Corrsin [31] (see also an excellent presentation in Ref. [30]). It has been shown by Weinstock [33] that the Corrsin approximation is the leading term in a systematic expansion and that the corrections to it are uniformly small if $\beta^2 \ll 1$. We adapted Weinstock's argument to our present problem and confirmed his conclusion (we do not publish here these very lengthy, but straightforward calculations). Furthermore, for a different but related problem [34], the consequences of the Corrsin approximation were checked by a numerical simulation [35].

In order to explain the Corrsin approximation, we write the Lagrangian correlation function as

$$\mathcal{L}_{mn}(\zeta)=\int d\hat{\boldsymbol{r}}_\perp\langle b_m(\hat{\boldsymbol{r}}_\perp,\zeta)b_n(\boldsymbol{x}_\perp(0),0)\delta(\hat{\boldsymbol{r}}_\perp-\boldsymbol{x}_\perp(\zeta))\rangle \; . \tag{24}$$

Corrsin assumes that, at least in some asymptotic sense, the exact propagator $\delta(\hat{\boldsymbol{r}}_\perp-\boldsymbol{x}_\perp(\zeta))$ can be approximated by its ensemble average, which then leads to a factorization of the integrand in Eq. (24), as follows:

$$\mathcal{L}_{mn}(\zeta)=\int d\hat{\boldsymbol{r}}_\perp\langle b_m(\hat{\boldsymbol{r}}_\perp,\zeta)b_n(\boldsymbol{x}_\perp(0),0)\rangle\langle\hat{\boldsymbol{r}}_\perp-\boldsymbol{x}_\perp(\zeta)\rangle\rangle$$
$$=\int d\boldsymbol{r}_\perp\langle b_m(\boldsymbol{r}_\perp,\zeta)b_n(0,0)\rangle\langle\delta(\hat{\boldsymbol{r}}_\perp-\delta\boldsymbol{x}_\perp(\zeta))\rangle \; , \tag{25}$$

where $\boldsymbol{r}_\perp=\hat{\boldsymbol{r}}_\perp-\boldsymbol{x}_\perp(0)$ and the notation $\delta\boldsymbol{x}_\perp(\zeta)=\boldsymbol{x}_\perp(\zeta)-\boldsymbol{x}_\perp(0)$ was introduced.

The Corrsin approximation amounts to establishing a functional relationship between the Lagrangian and the corresponding Eulerian correlation functions, in the form

$$\mathcal{L}_{mn}(\zeta)=\int d\boldsymbol{r}_\perp\gamma(\boldsymbol{r}_\perp,\zeta)\mathcal{B}_{mn}(\boldsymbol{r}_\perp,\zeta) \; , \tag{26}$$

where $\mathcal{B}_{mn}(\boldsymbol{r}_\perp,\zeta)\equiv\mathcal{B}_{mn}(r_x,r_y,\zeta)$ is the Eulerian field correlation, defined in Eq. (8) (with $r_z\rightarrow\zeta$). The function $\gamma(\boldsymbol{r}_\perp,\zeta)$ can be interpreted in this context as the probability of finding the current point on a field line at the perpendicular position $\boldsymbol{r}_\perp$ at "time" ζ, starting from $\boldsymbol{x}_\perp(0)$ at "time" 0. It can be represented as

$$\gamma(\boldsymbol{r}_\perp,\zeta)=\langle\delta(\boldsymbol{r}_\perp-\delta\boldsymbol{x}_\perp(\zeta))\rangle$$
$$=\int d\boldsymbol{k}_\perp e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{r}_\perp}\langle e^{-i\boldsymbol{k}_\perp\cdot\delta\boldsymbol{x}_\perp}\rangle \; . \tag{27}$$

It is easily shown that, as a result of the homogeneity and gyrotropy of the Eulerian potential correlations, the probability density has the property $\gamma(\boldsymbol{r}_\perp,\zeta)=\gamma(r_\perp,\zeta)$. Using this symmetry and Eq. (9) for the Eulerian correlations in Eq. (26), the following form is obtained for the Lagrangian correlations:

$$\mathcal{L}_{mn}(\zeta)=\delta_{mn}\mathcal{L}(\zeta) \; . \tag{28}$$

An explicit proof of this important property is given in the Appendix. In conclusion, we have shown that, for a homogeneous, stationary, and gyrotropic state, the Lagrangian correlation matrix is of the form (28), which is proportional to the unit matrix, with a single independent coefficient $\mathcal{L}(\zeta)$.

We now return to Eq. (27) and evaluate the average in the second cumulant approximation, with the MSD given by Eq. (22). Equation (28) is used and the $\boldsymbol{k}_\perp$ integral is performed, with the result

$$\gamma(r_\perp,\zeta)=\frac{1}{4\pi\int_0^\zeta d\zeta_1(\zeta-\zeta_1)\mathcal{L}(\zeta_1)}\times\exp\left[-\frac{r_\perp^2}{4\int_0^\zeta d\zeta_1(\zeta-\zeta_1)\mathcal{L}(\zeta_1)}\right] \; . \tag{29}$$

After introducing this expression into Eq. (26) and performing the space integral, we obtain an equation for $\mathcal{L}(\zeta)$:

$$\mathcal{L}(\zeta)=\beta^2\exp\left[-\frac{\zeta^2}{2\lambda_\parallel^2}\right]\frac{\lambda_\perp^4}{\left[\lambda_\perp^2+2\int_0^\zeta d\zeta_r(\zeta-\zeta_r)\mathcal{L}(\zeta_r)\right]^2} \; . \tag{30}$$

This is the basic integral equation obeyed by the Lagrangian correlation of the magnetic field fluctuations. Thus, even in the Corrsin approximation, the determination of the Lagrangian correlation requires the solution of a nonlinear integral equation, which cannot be obtained analytically.

It is interesting to note that in the limit $\lambda_\perp\rightarrow\infty$, Eq. (30) simplifies considerably: it becomes an explicit ex-

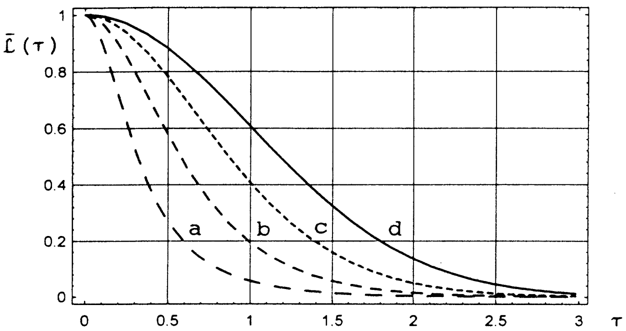

FIG. 1. Reduced Lagrangian correlation of magnetic field fluctuations. The curves have been obtained by numerical solution of Eq. (33). (a) $\alpha=2$, (b) $\alpha=1$, (c) $\alpha=\frac{1}{2}$, and (d) $\alpha=0$.

pression of $\mathcal{L}(\zeta)$. Comparing it with Eq. (17) we see that in this limit the Lagrangian correlation equals the Eulerian correlation:

$$\mathcal{L}(\zeta)=\beta^2\exp\left[-\frac{\zeta^2}{2\lambda_\parallel^2}\right]=\mathcal{B}(\zeta), \quad \lambda_\perp\to\infty \ . \tag{31}$$

For finite values of $\lambda_\perp$ Eq. (30) can be solved numerically. It is useful to introduce the dimensionless quantities

$$\tau=\frac{\zeta}{\lambda_\parallel}, \quad \alpha=\beta\frac{\lambda_\parallel}{\lambda_\perp}, \quad \bar{\mathcal{L}}(\tau)=\beta^{-2}\mathcal{L}(\lambda_\parallel\tau) \ . \tag{32}$$

The integral equation for $\bar{\mathcal{L}}(\tau)$ depends on the single parameter α,

$$\bar{\mathcal{L}}(\tau)=\frac{e^{-\tau^2/2}}{\left[1+2\alpha^2\int_0^\tau d\tau_1(\tau-\tau_1)\bar{\mathcal{L}}(\tau_1)\right]^2} \ . \tag{33}$$

The numerical solutions of this equation for various values of α are shown in Fig. 1. The Lagrangian correlation function will play a very important role in the theory of the magnetic line diffusion to be treated in the next section.

IV. MAGNETIC LINE DIFFUSION

As a result of the stochastic nature of the magnetic field, the field lines will (presumably) exhibit a diffusive behavior. We consider a representative (geometric) point on a (perturbed) field line starting at $\zeta=0$ on the average field line, i.e., $\delta x(0)=\delta y(0)=0$ [the notation $\delta\mathbf{x}(\zeta)$ was defined in Eq. (21)]. As we advance in the positive ζ direction, the representative point performs a random walk around the average line, according to Eq. (19). If the behavior is purely diffusive, the MSD $\Gamma(\zeta)$, defined in Eq. (22), behaves asymptotically (for large ζ) as a linear function [5–7,11,14,15,27]:

$$\Gamma(\zeta)=2D_m\zeta, \quad \zeta\to\infty \ , \tag{34}$$

where the constant D_m will be called the magnetic line diffusion coefficient: it has the dimension of a length. More generally we define, for all values of ζ, a running diffusion coefficient [27] as follows:

$$D_m(\zeta)=\frac{1}{2}\frac{d}{d\zeta}\Gamma(\zeta) \ , \tag{35}$$

with the property

$$\lim_{\zeta\to\infty} D_m(\zeta)=D_m \ . \tag{36}$$

If the behavior is diffusive, this constant is finite and different from zero.

It should be stated at this point that, in general, the MSD $\Gamma(\zeta)$ cannot be determined separately; rather, its defining equation couples it to the two other moments $\langle\delta y^2(\zeta)\rangle$ and $\langle\delta x(\zeta)\delta y(\zeta)\rangle$. It is shown, however, in the Appendix that under the conditions of stationarity, homogeneity, and gyrotropy, the MSD tensor reduces to a multiple of the unit tensor, whose only nonvanishing component is $\Gamma(\zeta)$. The MSD $\Gamma(\zeta)$ is simply related to the Lagrangian correlation of the magnetic fluctuations, through Eq. (22), from which it is easily found that

$$D_m(\zeta)=\int_0^\zeta d\zeta_1\mathcal{L}(\zeta_1) \ . \tag{37}$$

We now note that the integral equation (30) for the Lagrangian correlation function can be converted into an equation for the MSD. From Eqs. (35) and (37) we find the obvious relation $d^2\Gamma(\zeta)/d\zeta^2=2\mathcal{L}(\zeta)$; hence Eq. (30) is rewritten as

$$\frac{d^2\Gamma(\zeta)}{d\zeta^2}=2\beta^2\exp\left[-\frac{\zeta^2}{2\lambda_\parallel^2}\right]\frac{\lambda_\perp^4}{(\lambda_\perp^2+\Gamma(\zeta))^2} \ . \tag{38}$$

This is the basic differential equation for the MSD. It must be solved with the initial conditions

$$\Gamma(0)=0, \quad \left.\frac{d\Gamma(\zeta)}{d\zeta}\right|_{\zeta=0}=0 \ . \tag{39}$$

Note that these initial conditions are not arbitrary, but are determined by the initial forms (22) and (37) of the MSD and of D_m.

The differential equation (38) is nonlinear and cannot, in general, be solved analytically. In the limit $\lambda_\perp\to\infty$, the equation becomes integrable by quadratures, with the following simple result for the running diffusion coefficient [use Eq. (17)]:

$$D_{\rm QL}(\zeta)=\int_0^\zeta d\zeta_1\mathcal{B}(\zeta_1)=\left[\frac{\pi}{2}\right]^{1/2}\beta^2\lambda_\parallel\mathrm{erf}\left[\frac{\zeta}{\sqrt{2}\lambda_\parallel}\right] \ , \tag{40}$$

where erf(x) is the error function. From Eq. (36) we derive the magnetic line diffusion coefficient in this limit

$$D_{\rm QL}=\left[\frac{\pi}{2}\right]^{1/2}\beta^2\lambda_\parallel, \quad \lambda_\perp\to\infty \ . \tag{41}$$

This is a very well known expression of the magnetic line diffusion coefficient in the quasilinear limit [5,7,12]. It remains a good approximation also for finite $\lambda_\perp$, provided $\beta\lambda_\parallel/\lambda_\perp\ll 1$. The first nonlinear corrections to the quasilinear result can be obtained systematically by a perturbation procedure. Introducing the dimensionless quantities defined in Eq. (32) as well as the dimensionless MSD $g(\tau)=\Gamma(\lambda_{\parallel}\tau)/\lambda_{\perp}^2$, the differential equation is rewritten as

$$\frac{d^2g(\tau)}{d\tau^2}=2\alpha^2 e^{-\tau^2/2}\frac{1}{[1+g(\tau)]^2} . \tag{42}$$

This equation can be solved by a perturbation method, assuming $\alpha^2 \ll 1$. The calculation is standard (although the term of order α^6 requires working out some rather complicated integrals involving error functions) and leads to the following dimensionless magnetic line diffusion coefficient $\mathcal{D}_m=(\lambda_{\parallel}/\lambda_{\perp}^2)D_m$:

$$\mathcal{D}_M=\left[\frac{\pi}{2}\right]^{1/2}\alpha^2\left[1-4(\sqrt{2}-1)\alpha^2 +4\left[7\sqrt{3}-10\sqrt{2}+3+\frac{\pi}{6}\right]\alpha^4\right] . \tag{43}$$

The leading term $\mathcal{D}_m=(\pi/2)^{1/2}\alpha^2$, when written in dimensional form, is, of course, equivalent to Eq. (41). The first effect of the nonlinearity is a lowering of the diffusion coefficient.

The opposite limit is very interesting, although it is physically not very realistic: it is the case $\lambda_{\parallel}\to\infty$, $\lambda_{\perp}$ finite. Equation (38) then reduces to

$$\frac{d^2\Gamma(\zeta)}{d\zeta^2}=2\beta^2\frac{\lambda_{\perp}^4}{[\lambda_{\perp}^2+\Gamma(\zeta)]^2} . \tag{44}$$

Using a standard method, this equation is integrated, with the following result for the dimensionless diffusion coefficient:

$$\frac{dg(\tau)}{d\tau}\equiv 2\mathcal{D}_{\infty}(\tau)=2\alpha\left[\frac{g(\tau)}{1+g(\tau)}\right]^{1/2} . \tag{45}$$

This is the dimensionless form of the running diffusion coefficient. Its asymptotic limit is found by letting $\tau\to\infty$ and hence $g(\tau)\to\infty$:

$$\mathcal{D}_{\infty}=\alpha . \tag{46}$$

The corresponding dimensional form is

$$D_{\infty}=\beta\lambda_{\perp} . \tag{47}$$

This result for the magnetic line diffusion coefficient in the extreme percolation limit $\lambda_{\parallel}\to\infty$ was previously obtained in Ref. [7] in a semiqualitative way. Here all the limiting results are obtained as particular cases of the solution of a unique differential equation (38). The very conspicuous feature of the result (47) is the linear dependence on β, to be contrasted with the quadratic dependence in the quasilinear limit. This is very much reminiscent of the behavior of the particle diffusion coefficient as a function of the fluctuation intensity η of electrostatic drift waves [34–39]. In that case one sees a similar transition from a quadratic dependence (for small η) to a linear dependence (for large η): the latter is called the Bohm-like regime. The numerical solution of the complete equation (38), plotted in Fig. 2, clearly shows the transition from the quadratic to the linear regime.

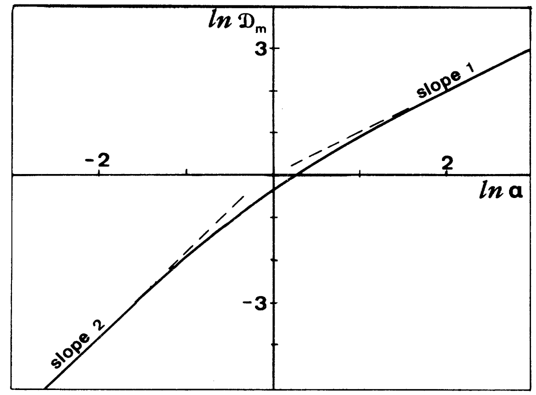

FIG. 2. Dimensionless magnetic line diffusion coefficient $\mathcal{D}_m(\alpha)$. The full curve was obtained by numerical integration of Eq. (42). The diffusion coefficient was determined from the limiting value of $dg(\tau)/d\tau$ for large values of τ.

The Kadomtsev-Pogutse [7] $\lambda_{\parallel}\to\infty$ limit (47), confirmed here as the corresponding limiting solution of Eq. (38) is, however, incorrect. Indeed, large values of $\lambda_{\parallel}$, and hence of α, are outside the domain of validity of Eq. (38) itself (not only of its solutions). In particular, the validity of the Corrsin approximation (26) is only ascertained for $\beta \ll 1$, and hence $\alpha \ll 1$. The problem of the magnetic line diffusion must actually be treated by quite different methods, taken from percolation theory [15,24], which will not be discussed further here.

V. RELATIVE DISTANCE OF NEIGHBORING MAGNETIC FIELD LINES

In the preceding section we considered the diffusive motion of a single field line. We now consider two magnetic field lines $\boldsymbol{x}_{\perp 1}(\zeta),\boldsymbol{x}_{\perp 2}(\zeta)$, starting at different positions at $\zeta=0$. We introduce the notation $\Delta\boldsymbol{x}_{\perp}(\zeta)=[\Delta x(\zeta),\Delta y(\zeta)]$ to denote the instantaneous distance between the two orbits at "time" ζ:

$$\Delta\boldsymbol{x}_{\perp}(\zeta)=\boldsymbol{x}_{\perp 2}(\zeta)-\boldsymbol{x}_{\perp 1}(\zeta) . \tag{48}$$

The equation of evolution for this quantity is easily derived from Eq. (19) and, for small values of $|\Delta\boldsymbol{x}_{\perp}(\zeta)|$, linearized around $\boldsymbol{x}_{\perp 1}(\zeta)$ [see also Refs. [11,12,14]:

$$\frac{d\Delta x(\zeta)}{d\zeta}=b_x(\boldsymbol{x}_2(\zeta),\zeta)-b_x(\boldsymbol{x}_1(\zeta),\zeta)$$
$$=b_{x,x}(\zeta)\Delta x(\zeta)+b_{x,y}(\zeta)\Delta y(\zeta) ,$$

where $b_{m,n}(\zeta)$ denotes the partial derivatives of the magnetic field, evaluated at the instantaneous position $\boldsymbol{x}_{\perp 1}(\zeta)$:

$$b_{m,n}(\zeta)=\frac{\partial}{\partial x_n}b_m(\boldsymbol{x}_{\perp},\zeta)\Big|_{\boldsymbol{x}_{\perp}=\boldsymbol{x}_{\perp}(\zeta)} . \tag{49}$$

Introducing now the matrix $\underline{B}(\zeta)\equiv[b_{m,n}(\zeta)]$, we write

$$\frac{d}{d\zeta}\Delta \mathbf{x}_\perp(\zeta)=\underline{B}(\zeta)\cdot\Delta \mathbf{x}_\perp(\zeta) . \tag{50}$$

The matrix $\underline{B}$ has zero trace (because of the zero-divergence constraint $b_{n,n}=0$); hence its eigenvalues have equal absolute values and opposite sign:

$$\lambda_{+,-}=\pm\sqrt{b_{x,x}^2+b_{x,y}b_{y,x}} . \tag{51}$$

Consequently, one of the two eigenvectors grows exponentially with ζ and the other one decreases at the same rate. This Lyapounov stability analysis, valid for a given realization of the magnetic field, shows that, at each point (x,y), there is a tendency toward exponential separation of the two lines in one direction and of the exponential approach in another direction. It should be stressed, however, that this statement only holds locally: even the linearized equations of motion do not possess simple exponential solutions of the type $\exp(\lambda_i\zeta)$, because the coefficients $b_{m,n}(\zeta)$ are not constant, but rather ζ dependent. These coefficients determine the local rates and directions of exponentiation (unstable and stable manifolds); they vary from one point to its neighbor. As a result, a small initial circle in the (x,y) plane transforms upon advancing along ζ into a complicated pattern with longer and longer boundary and invariant area (see Isichenko's "flower" in Ref. [14]).

We now consider the truly statistical problem in which $\mathbf{b}$ is considered as a random field and the equations of motion (50) must be treated as stochastic differential equations. The statistical description of the relative evolution of a pair of field lines requires the determination of (at least) three moments $\langle \Delta x^2(\zeta)\rangle$, $\langle \Delta y^2(\zeta)\rangle$, and $\langle \Delta x(\zeta)\Delta y(\zeta)\rangle$. This problem was treated previously in Refs. [11,12,14]; we therefore omit the details of the derivation. A set of three equations for these moments is derived from Eq. (50); their coefficients involve the Lagrangian correlations of the magnetic field gradients $\mathcal{L}_{mn}^{\alpha\beta}(\zeta)$ corresponding to the Eulerian ones of Eq. (13). Generalizing the calculations of Sec. III, these quantities are evaluated with a Corrsin approximation similar to Eq. (26). It is then shown that the Eulerian correlations (13) imply that all the Lagrangian correlations of the field gradients are expressible in terms of a single scalar function $\mathcal{K}(\zeta)$,

$$\begin{aligned}\mathcal{L}_{xx}^{xx}(\zeta)=\mathcal{L}_{yy}^{yy}(\zeta)&=\tfrac{1}{3}\mathcal{L}_{xx}^{yy}(\zeta)\\ &=\tfrac{1}{3}\mathcal{L}_{yy}^{xx}(\zeta)=-\mathcal{L}_{xy}^{xy}(\zeta)\\ &=-\mathcal{L}_{xy}^{yx}(\zeta)=\mathcal{K}(\zeta) .\end{aligned} \tag{52}$$

These properties are analogous to the result (28). By the same arguments as in Sec. III, it is shown that $\mathcal{K}(\zeta)$ is expressed in terms of the function $\mathcal{L}(\zeta)$ defined in Eq. (28) as

$$\mathcal{K}(\zeta)=\beta^2\exp\left[-\frac{\zeta^2}{2\lambda_\parallel^2}\right]\frac{\lambda_\perp^4}{\left[\lambda_\perp^2+2\int_0^\zeta d\zeta_1(\zeta-\zeta_1)\mathcal{L}(\zeta_1)\right]^3} . \tag{53}$$

After some elementary algebra, the equations for the relative MSD's, treated asymptotically for $\zeta\gg\lambda_\parallel$ (the Markovian approximation), reduce to the simplified form

$$\begin{aligned}\frac{d}{d\zeta}\langle \Delta x^2(\zeta)\rangle&=2\mathcal{K}\langle \Delta x^2(\zeta)\rangle+6\mathcal{K}\langle \Delta y^2(\zeta)\rangle ,\\ \frac{d}{d\zeta}\langle \Delta y^2(\zeta)\rangle&=6\mathcal{K}\langle \Delta x^2(\zeta)\rangle+2\mathcal{K}\langle \Delta y^2(\zeta)\rangle ,\\ \frac{d}{d\zeta}\langle \Delta x(\zeta)\Delta y(\zeta)\rangle&=-4\mathcal{K}\langle \Delta x(\zeta)\Delta y(\zeta)\rangle ,\end{aligned} \tag{54}$$

where

$$\mathcal{K}\equiv\frac{1}{4L_K}=\int_0^\infty d\zeta\,\mathcal{K}(\zeta) . \tag{55}$$

The quantity L_K defined by this equation has the dimension of a length: it will be called the exponentiation length for reasons that will presently be clear. The solution of the corresponding initial value problem (assuming deterministic initial values) is found by standard methods:

$$\begin{aligned}\langle \Delta x^2(\zeta)\rangle&=\tfrac{1}{2}[\Delta x^2(0)+\Delta y^2(0)]\exp\left[2\frac{\zeta}{L_K}\right]\\ &\quad+\tfrac{1}{2}[\Delta x^2(0)-\Delta y^2(0)]\exp\left[-\frac{\zeta}{L_K}\right] ,\\ \langle \Delta y^2(\zeta)\rangle&=\tfrac{1}{2}[\Delta x^2(0)+\Delta y^2(0)]\exp\left[2\frac{\zeta}{L_K}\right]\\ &\quad-\tfrac{1}{2}[\Delta x^2(0)-\Delta y^2(0)]\exp\left[-\frac{\zeta}{L_K}\right] ,\\ \langle \Delta x(\zeta)\Delta y(\zeta)\rangle&=\Delta x(0)\Delta y(0)\exp\left[-\frac{\zeta}{L_K}\right] .\end{aligned} \tag{56}$$

In the limit of a very large perpendicular correlation length, the integral in the denominator of Eq. (53) can be neglected and one finds a Gaussian form for the Lagrangian correlation which, combined with Eq. (55), yields a very simple form for the exponentiation length L_K,

$$L_K=\frac{\lambda_\perp^2}{4D_{\mathrm{QL}}}=\left[\frac{2}{\pi}\right]^{1/2}\frac{\lambda_\perp^2}{4\beta^2\lambda_\parallel} , \tag{57}$$

where D_{QL} is the quasilinear magnetic line diffusion coefficient (41). Clearly, L_K is the characteristic length scale of the moments $\langle \Delta x^2(\zeta)\rangle$, etc. It can be shown that the criterion of validity of the Markovian approximation, and hence of Eqs. (54) and (56), requires the parallel correlation length $\lambda_\parallel$ to be much smaller than the exponentiation length

$$\frac{\lambda_\parallel}{L_K}=4\left[\frac{\pi}{2}\right]^{1/2}\beta^2\frac{\lambda_\parallel^2}{\lambda_\perp^2}\ll 1 . \tag{58}$$

This criterion is easily satisfied in realistic situations. We now see that the two MSD's $\langle \Delta x^2\rangle,\langle \Delta y^2\rangle$ have an exponentially growing and an exponentially decaying part, both on a scale of order L_K. The cross correlation $\langle \Delta x(\zeta)\Delta y(\zeta)\rangle$ is decaying in ζ; the two MSD's can be combined as

$$\langle \Delta x^2(\zeta) \rangle + \langle \Delta y^2(\zeta) \rangle \equiv \langle \Delta r_\perp^2(\zeta) \rangle = \Delta r_\perp^2(0) \exp\left[2\frac{\zeta}{L_K}\right], \tag{59}$$

$$\langle \Delta x^2(\zeta) \rangle - \langle \Delta y^2(\zeta) \rangle = [\Delta x^2(0) - \Delta y^2(0)] \exp\left[-\frac{\zeta}{L_K}\right].$$

This result is extremely interesting. It shows that in the average picture the distance between the lines grows exponentially and gyrotropically in all directions. The possible initial differences $\Delta x(0) \neq \Delta y(0)$ are progressively washed out; there is a loss of memory of any initial privileged direction.

It should be realized that the exponentiation of the magnetic lines only represents an initial trend of the evolution. Indeed, the solution (59) is based on the linearized equations (54). Whenever the small initial separation becomes sufficiently large, the nonlinear effects come into play and seriously modify the picture; in particular the exponential growth is slowed down (presumably to a diffusive growth, linear in ζ). This situation is strongly reminiscent of the clump effect studied in electrostatic turbulence [34,40]. The nonlinear study of this problem is left for a forthcoming paper.

Our results will now be compared with those of previous workers. We first note that there is no intersection with the work of Krommes, Oberman, and Kleva [11]. These authors used a sheared reference magnetic field, characterized by a finite shear length L_s; their calculation automatically yields infinite exponentiation lengths whenever $L_s \to \infty$. Isichenko [14] also has a sheared reference field in his model, but his more careful calculation shows that the exponentiation lengths remain finite even in the limit of zero shear. More precisely, he shows that the result of Ref. [11] can be obtained as a limit when $L_s \ll L_K$.

In the opposite limit $L_s \gg L_K$, Isichenko's results should reduce to ours: actually, they do not. We already pointed out in Sec. II that his statistical assumptions about the magnetic fluctuations are inappropriate, as they are inconsistent with the constraint of zero divergence of the magnetic field. In order to exhibit the consequences of this feature, we define, as in Refs. [11,14], a vector $\boldsymbol{R} = (\langle \Delta x^2 \rangle, \langle \Delta y^2 \rangle, \langle \Delta x \Delta y \rangle)$ and write the linearized and Markovian equations (54) in the form

$$\frac{d}{d\zeta}\boldsymbol{R} = \underline{A} \cdot \boldsymbol{R}, \tag{60}$$

with

$$\underline{A} = \frac{1}{2L_K}\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{61}$$

This is to be compared with the following matrix obtained by Isichenko (for $b_z = 0$ and $L_s = \infty$):

$$\underline{A}_{\text{Isi}} = \frac{1}{2L_K}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{62}$$

The fact that the numerical coefficients are slightly different in the two cases is not too disturbing. There is, however, a much more serious qualitative difference. The eigenvalues of our matrix are the coefficients appearing in the arguments of the exponentials in Eq. (56):

$$\frac{2}{L_K}, \quad -\frac{1}{L_K}, \quad -\frac{1}{L_K}, \quad \text{eigenvalues of } \underline{A}. \tag{63}$$

The corresponding quantities in Ref. [14] are

$$\frac{6}{L_K}, \quad \frac{2}{L_K}, \quad 0, \quad \text{eigenvalues of } \underline{A}_{\text{Isi}}. \tag{64}$$

Instead of one positive and two negative exponents, Isichenko finds two positive and one vanishing exponent. There is therefore an indiscriminate growth of the mean square displacements in all directions, whereas the cross correlations remain constant. The important features, and in particular the gyrotropization of the state described above, are missed in Ref. [14].

Another interesting consequence of Eqs. (56) and (59) is the existence of a constant of the motion (in the linearized approximation)

$$\begin{aligned}\mathcal{F}(\zeta) \equiv \{ &[\langle \Delta x^2(\zeta) \rangle + \langle \Delta y^2(\zeta) \rangle] \\ &\times [\langle \Delta x^2(\zeta) \rangle - \langle \Delta y^2(\zeta) \rangle] \\ &\times \langle \Delta x(\zeta) \Delta y(\zeta) \rangle \} = \mathcal{F}(0).\end{aligned} \tag{65}$$

This feature is clearly a consequence of the vanishing trace of the matrix $\underline{A}$ or, equivalently, of the fact that its eigenvalues sum up to zero, Eq. (63). This property could not have been obtained by Isichenko. All these differences arise from the fact that he does not take into account the existence of nondiagonal Eulerian correlations of the field components, which are imposed by the condition $\nabla \cdot \boldsymbol{b} = 0$.

VI. DECORRELATION OF PARTICLES FROM FIELD LINES

We now introduce a plasma into the previously described magnetic field configuration. We thus go over from the kinematic (geometric) description to the dynamic study of the motion of charged particles under the action of the magnetic field and of their mutual collisions. In order to make the problem tractable, we assume that the unperturbed field B_0 is very strong; the motion of the particles can then be described in the drift approximation. Neglecting all finite Larmor radius effects, we assimilate the position of the particles with the position of their guiding centers. The latter are supposed to move along the (perturbed) magnetic field; the perpendicular drift motions are neglected (the effects of the latter will be considered in a forthcoming paper). The collisions are modeled by a random velocity with a component $\eta_\parallel$ in the z direction and components $\eta_{\perp x}, \eta_{\perp y}$ in the perpendicular direction. As a result, the equations of motion of the particles are obtained by combining the field line equations with the collisional velocity

$$\frac{d}{dt}x_p(t) = b_x[x_p(t), y_p(t), z_p(t)]\frac{dz_p(t)}{dt} + \eta_{\perp x}(t), \tag{66}$$

$$\frac{d}{dt}y_p(t)=b_y[x_p(t),y_p(t),z_p(t)]\frac{dz_p(t)}{dt}+\eta_{\perp y}(t) , \quad (67)$$

$$\frac{d}{dt}z_p(t)=\eta_{\parallel}(t) . \quad (68)$$

These are the so-called V-Langevin equations. They were derived and used in many previous works [7,11,14,18]; a thorough discussion is given in Ref. [27]. In Eqs. (66)–(68), $[x_p(t),y_p(t),z_p(t)]$ are the coordinates of the instantaneous position of a particle (or guiding center) at time t. This point of view is to be distinguished from the quantities entering Eq. (19). In particular, $[x(\zeta),y(\zeta),\zeta]$ in the latter equation denotes the coordinates x and y of a geometrical point on a field line, parametrized by the third spatial coordinate ζ. In the present dynamical picture, the latter becomes a function of time $\zeta \to z_p(t)$; as a result, $x[z_p(t),t] \to x_p(t), y[z_p(t),t] \to y_p(t)$. Equations (66)–(68) must be completed by a statistical definition of the random velocities. We assume that $\eta_{\perp x},\eta_{\perp y},\eta_{\parallel}$ have zero average and are modeled by a Gaussian colored noise, with identical statistical properties for $\eta_{\perp x},\eta_{\perp y}$:

$$\langle \eta_{\parallel}(t)\eta_{\parallel}(t')\rangle=\chi_{\parallel}\nu \exp[-\nu|t-t'|]\equiv R_{\parallel}(|t-t'|) , \quad (69)$$

$$\begin{aligned}\langle \eta_{\perp x}(t)\eta_{\perp x}(t')\rangle &=\langle \eta_{\perp y}(t)\eta_{\perp y}(t')\rangle\\ &=\chi_{\perp}\nu \exp[-\nu|t-t'|]\\ &\equiv R_{\perp}(|t-t'|) , \end{aligned} \quad (70)$$

$$\begin{aligned}\langle \eta_{\perp x}(t)\eta_{\perp y}(t')\rangle &=\langle \eta_{\perp x}(t)\eta_{\parallel}(t')\rangle\\ &=\langle \eta_{\perp y}(t)\eta_{\parallel}(t')\rangle=0 . \end{aligned} \quad (71)$$

Comparing these results with transport theory [27,28] we interpret ν as the collision frequency of the plasma and $\chi_{\parallel},\chi_{\perp}$ as the classical (collisional) diffusion coefficients parallel or perpendicular, respectively, to the strong magnetic field $B_0 e_z$. In terms of the thermal velocity $V_T=\sqrt{2T/m}$ (where m is the mass of a particle and T the temperature of the medium), these transport coefficients have the following values [27]:

$$\chi_{\parallel}=\frac{V_T^2}{2\nu},\ \ \chi_{\perp}=\frac{\nu}{2\Omega^2}V_T^2,\ \ \frac{\chi_{\perp}}{\chi_{\parallel}}=\left[\frac{\nu}{\Omega}\right]^2 \ll 1 . \quad (72)$$

Here $\Omega=eB_0/mc$ is the (unperturbed) Larmor frequency of the test particle of charge e. It is well known that in a strong magnetic field, the perpendicular diffusion coefficient is much smaller than the parallel one.

The V-Langevin equations (66)–(68) describe a triply stochastic process. The first random process is the stochastic magnetic field, which produces chaos of the field lines as described in Sec. V, and an asymptotic diffusion of the latter, as shown in Sec. IV. The parallel collisional velocity $\eta_{\parallel}$ produces a diffusion of the particles along the magnetic field lines and $\eta_{\perp}$ produces a departure from the field lines. The result of the combination of these three random factors in the V-Langevin equations leads to a very difficult problem that has never been solved exactly. In the present paper we address a specific question: How can the collisions produce a decorrelation of the particles from the magnetic field lines? If such a decorrelation mechanism exists, the particles can get loose from the lines to which they were initially tied and diffuse across the magnetic field much more strongly than predicted by the purely collisional mechanism (in the absence of magnetic fluctuations).

In order to study this problem we introduce the following concepts. Consider a particle located at time $t=0$ at a point $(x_p(0),y_p(0),z_p(0))$ and let $(x_m(\zeta),y_m(\zeta))$ represent the magnetic field line passing through the position of the particle at time zero (briefly, the initial field line), i.e., $x_m(\zeta_0)=x_p(0)$ and $y_m(\zeta_0)=y_p(0)$ for $\zeta_0=z_p(0)$. At time $t>0$, the particle has left the field line; therefore the perpendicular component of its position can be represented as

$$\boldsymbol{x}_{p\perp}(t)=\boldsymbol{x}_{m\perp}(z_p(t))+\Delta \boldsymbol{x}_{p\perp}(t) , \quad (73)$$

where $\boldsymbol{x}_{m\perp}(z_p(t))$ is the position where the particle would be at time t had it followed the initial field line and $\Delta\boldsymbol{x}_{p\perp}(t)$ is the deviation, at time t, of the real particle position from the fictitious position on the initial field line (Fig. 3). The equations for the perpendicular components of the deviation $\Delta\boldsymbol{x}_{p\perp}(t)$ are obtained from the V-Langevin equations (66)–(68), which define the trajectory and Eq. (19) for the field lines:

$$\begin{aligned}\frac{d}{dt}\Delta\boldsymbol{x}_{p\perp}(t)=\{&\boldsymbol{b}(x_p(t),y_p(t),z_p(t))\\ &-\boldsymbol{b}(x_m(z_p(t)),y_m(z_p(t)),z_p(t))\}\\ &\times\frac{dz_p(t)}{dt}+\eta_{\perp}(t) .\end{aligned}$$

For $|\Delta x_p(t)|,|\Delta y_p(t)| \ll \lambda_{\perp}$, these equations can be linearized as

$$\begin{aligned}\frac{d}{dt}\Delta x_p(t)=\,&b_{x,x}(\boldsymbol{x}_{m\perp}(z_p(t)),z_p(t))\frac{dz_p(t)}{dt}\Delta x_p(t)\\ &+b_{x,y}(\boldsymbol{x}_{m\perp}(z_p(t)),z_p(t))\frac{dz_p(t)}{dt}\Delta y_p(t)\\ &+\eta_{\perp x}(t)\end{aligned} \quad (74)$$

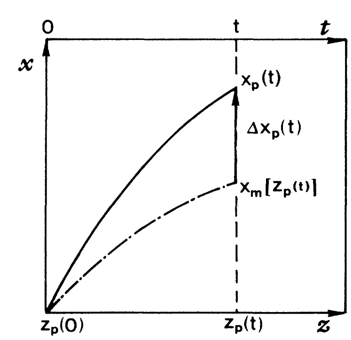

FIG. 3. Definition of the particle deviation. The full line is a particle orbit, intersecting at time $t=0$ the magnetic field line represented by the dash-dotted line.

and a similar equation for $\Delta y_p(t)$. The equations must be solved with the initial conditions $\Delta x_p(0)=0$, $\Delta y_p(0)=0$, and $z_p(0)=0$.

These equations are very similar to Eq. (50) for the field line separation. They can therefore be treated by a similar method. One first derives equations for the MSD between particle and initial field line, as well as the cross correlations, averaged over the ensemble of magnetic fluctuations (in the present triply stochastic problem the type of averaging must be specified and introduced as a subscript in the notations). The equations are treated with the same approximations as in Sec. V, i.e., the Corrsin and the Markovian approximations:

$$\frac{d}{dt}\langle \Delta x_p^2(t)\rangle_b = 2\mathcal{K}\frac{dz_p(t)}{dt}\langle \Delta x_p^2(t)\rangle_b + 6\mathcal{K}\frac{dz_p(t)}{dt}\langle \Delta y_p^2(t)\rangle_b + 2\int_0^t d\tau\,\eta_{\perp x}(t)\eta_{\perp x}(\tau)\ , \tag{75}$$

$$\frac{d}{dt}\langle \Delta y_p^2(t)\rangle_b = 6\mathcal{K}\frac{dz_p(t)}{dt}\langle \Delta x_p^2(t)\rangle_b + 2\mathcal{K}\frac{dz_p(t)}{dt}\langle \Delta y_p^2(t)\rangle_b + 2\int_0^t d\tau\,\eta_{\perp y}(t)\eta_{\perp y}(\tau)\ , \tag{76}$$

$$\frac{d}{dt}\langle \Delta x_p(t)\Delta y_p(t)\rangle_b = -4\mathcal{K}\frac{dz_p(t)}{dt}\langle \Delta x_p(t)\Delta y_p(t)\rangle_b + \int_0^t d\tau[\eta_{\perp x}(t)\eta_{\perp y}(\tau) + \eta_{\perp y}(t)\eta_{\perp x}(\tau)]\ , \tag{77}$$

where $\mathcal{K}=(4L_K)^{-1}$ was defined in Eq. (55).

The second averaging of these equations over the ensemble of perpendicular random velocities $\eta_\perp(t)$ is immediate, using Eq. (70). This set of equations is similar to Eqs. (54), up to the source term due to the perpendicular collisions. The solution of Eqs. (75)–(77) is obtained by standard methods:

$$\langle \Delta x_p^2(t)\rangle_{b,\perp} = \langle \Delta y_p^2(t)\rangle_{b,\perp} = 2\int_0^t dt_1\int_0^{t_1} d\tau\, R_\perp(t_1-\tau) \times\exp\{8\mathcal{K}[z_p(t)-z_p(t_1)]\}\ ,$$
$$\langle \Delta x_p(t)\Delta y_p(t)\rangle_{b,\perp}=0\ . \tag{78}$$

All these results were obtained in a given realization of $z_p(t)$ [or of $\eta_\parallel(t)$]. They must finally be averaged over the latter random quantity. We now put these results together in an expression of the absolute MSD of the particle position:

$$\langle \delta x_p^2(t)\rangle_{b,\perp,\parallel} = \langle \delta x_m^2(z_p(t))\rangle_{b,\parallel} + \langle \Delta x_p^2(t)\rangle_{b,\perp,\parallel} \tag{79}$$

[with a similar formula for $\langle \delta y_p^2(t)\rangle$]. The decomposition (79) is a very specific feature of our method. From this equation for the MSD we obtain a corresponding decomposition of the running diffusion coefficient

$$D(t) \equiv \tfrac{1}{2}\partial_t\langle \delta x_p^2(t)\rangle_{b,\perp,\parallel} = D^{(1)}(t)+D^{(2)}(t) \tag{80}$$

[where the superscripts (1) and (2) refer to the first and the second terms of Eq. (79), respectively] and a similar decomposition for the limiting diffusion coefficient

$$D \equiv \lim_{t\to\infty} D(t) = D^{(1)}+D^{(2)}\ . \tag{81}$$

The first term was obtained in the theory of magnetic line diffusion, Eq. (22): it represents the lateral displacement of particles sticking to the field lines and dragged along by the diffusion of these lines. The second term is given by Eq. (78): it represents the perpendicular displacement produced by a decorrelation of the particles from the field lines. Note that the latter term is the only one containing the perpendicular diffusion coefficient $\chi_\perp$ and is proportional to it [see Eq. (70)]. Hence the first term of Eq. (79) can also be interpreted as the value of the MSD for $\chi_\perp=0$.

We now perform the average over the parallel velocity $\eta_\parallel(t)$ explicitly. The following results will be used in this operation:

$$\langle z_p(t)\rangle_\parallel = 0\ ,$$
$$\langle z_p^2(t)\rangle_\parallel = 2\int_0^t d\tau\, R_\parallel(\tau)(t-\tau) = 2\frac{\chi_\parallel}{\nu}\psi(\nu t)\ , \tag{82}$$

$$\psi(x) = x-1+e^{-|x|}\ . \tag{83}$$

A. The first term in Eq. (79)

The first term in Eq. (79) is evaluated by using Eq. (22) in the form

$$\langle \delta x_m^2(t)\rangle_{b,\parallel} = \left\langle \int_0^z d\zeta_1\int_{-\zeta_1}^{z-\zeta_1} d\zeta\,\mathcal{L}(\zeta)\right\rangle_\parallel = \int_{-\infty}^{\infty} dk\,\mathcal{L}(k)k^{-2}\langle 2-e^{ikz}-e^{-ikz}\rangle_\parallel\ , \tag{84}$$

where $z\equiv z_p(t)$, $\mathcal{L}(\zeta)$ is the scalar function characterizing the Lagrangian magnetic field correlations, Eq. (28), and $\mathcal{L}(k)$ is its Fourier transform. The average of the exponentials is calculated in the second cumulant approximation and, after some algebra, one obtains

$$\langle \delta x_m^2(t)\rangle_{b,\parallel} = \int_{-\infty}^{\infty} d\zeta\langle \delta x_m^2(\zeta)\rangle_b P(\zeta,t)\ , \tag{85}$$

where

$$P(\zeta,t) = [2\pi\langle z_p^2(t)\rangle_\parallel]^{-1/2}\exp\left\{-\frac{\zeta^2}{2\langle z_p^2(t)\rangle_\parallel}\right\}\ . \tag{86}$$

This function can clearly be interpreted as the probability density for a particle to be located, at time t, at the position corresponding to coordinate ζ on the magnetic line. Equation (85) thus shows that, in the absence of perpendicular collisions, the cross-field particle displacement is expressed as the average of the magnetic line displacement, weighted by the distribution function $P(\zeta,t)$. Equation (85) yields the following expression for $D^{(1)}(t)$, Eq. (80):

$$D^{(1)}(t)=\varphi(\nu t)\left[\frac{\chi_\parallel \nu}{\pi\psi(\nu t)}\right]^{1/2}\times\int_0^\infty d\zeta\,\mathcal{L}(\zeta)\exp\left[-\frac{\zeta^2}{4(\chi_\parallel/\nu)\psi(\nu t)}\right], \quad (87)$$

where $\varphi(x)=1-e^{-x}$.

For long times $\nu t\gg 1$ [such that $\langle z_p^2(t)\rangle \gg \lambda_\parallel^2$], the exponential in the integrand of Eq. (87) can be approximated by 1 [because $\psi(\nu t)\sim \nu t$] and the asymptotic form of the function $D^{(1)}(t)$ is

$$D^{(1)}(t)\sim\left[\frac{\chi_\parallel}{\pi t}\right]^{1/2}\int_0^\infty d\zeta\,\mathcal{L}(\zeta) \quad (88)$$

or, using the definition of the magnetic line diffusion coefficient, Eq. (37),

$$D^{(1)}(t)=\frac{1}{\sqrt{\pi}}\sqrt{\chi_\parallel}D_m\frac{1}{\sqrt{t}},\quad \nu t\gg 1,\quad \chi_\perp=0 . \quad (89)$$

Thus the running diffusion coefficient is proportional to the magnetic line diffusion coefficient and to the square root of the parallel collision diffusion coefficient.

The most important conclusion to be drawn from Eq. (89) is the following. For all values of the perpendicular correlation length (and not only for $\lambda_\perp\to\infty$), whenever $\chi_\perp=0$ and when the perpendicular drift motions can be neglected, the asymptotic behavior of the particles is subdiffusive, with $D^{(1)}(t)\sim t^{-1/2}$ and $D^{(1)}=0$.

It should be stressed, however, that this conclusion is only valid for collisional plasmas, when $\nu\neq 0$. The collisionless limit of Eq. (87) is rather singular. When $\nu\to 0$ (for fixed t), we set, in Eq. (87), $\varphi(\nu t)\sim \nu t$ and $\psi(\nu t)\sim\frac{1}{2}(\nu t)^2$, use Eq. (72) $\chi_\parallel\nu=V_T^2/2$, and consider, for definiteness, the approximation (31) for $\mathcal{L}(\zeta)$:

$$D_0^{(1)}(t)=\pi^{-1/2}V_T\beta^2\int_0^\infty d\zeta\exp\left[-\frac{1}{2}\left[\frac{1}{\lambda_\parallel^2}+\frac{1}{V_T^2t^2}\right]\zeta^2\right]$$
$$=2^{-1/2}V_T\beta^2\left[\frac{1}{\lambda_\parallel^{-2}+(V_Tt)^{-2}}\right],\quad \nu=0 . \quad (90)$$

This running diffusion coefficient tends to a nonzero limit as $t\to\infty$,

$$D_0^{(1)}=2^{-1/2}\beta^2\lambda_\parallel V_T=\pi^{-1/2}D_{\rm QL}V_T,\quad \nu=0 . \quad (91)$$

Here $D_{\rm QL}$ is the magnetic line diffusion coefficient (in the quasilinear approximation) defined in Eq. (41). Thus, in the strictly collisionless limit, the first term of Eq. (79) behaves diffusively. This result will be discussed further below.

B. The second term in Eq. (79)

We now consider the $\eta_\parallel$ averaging of the second term in Eq. (79), i.e., the term describing the decorrelation produced by the perpendicular collisional diffusion coefficient, Eq. (78). The average of the exponential is evaluated in the second cumulant approximation

$$\left\langle \exp\left[\frac{2}{L_K}[z_p(t)-z_p(t_1)]\right]\right\rangle_\parallel=\exp\left[\frac{4\chi_\parallel}{L_K^2\nu}\psi[\nu(t-t_1)]\right] . \quad (92)$$

Here the following dimensionless parameter appears:

$$\mu=\frac{4\chi_\parallel}{\nu L_K^2}=2\left[\frac{L_{\rm mfp}}{L_K}\right]^2 , \quad (93)$$

where $L_{\rm mfp}=V_T/\nu$ is the collisional mean free path. Clearly, a small value of μ corresponds to a strongly collisional regime. Using Eqs. (92) and (70), Eq. (78) is now written in the form

$$\langle \Delta x_p^2(t)\rangle_{b,\perp,\parallel}=2\chi_\perp\int_0^t dt_1\varphi[\nu(t-t_1)]\exp[\mu\psi(\nu t_1)] . \quad (94)$$

This expression will be evaluated in the two limiting cases of strong and weak collisionality.

1. The limit of strong collisionality

For large t we approximate $\psi(\nu t)\sim \nu t$ and obtain

$$\langle \Delta x_p^2(t)\rangle_{b,\perp,\parallel}=\frac{2\chi_\perp}{\mu\nu}\left[\frac{e^{\mu\nu t}}{1+\mu}-1\right] , \quad (95)$$

which, in the strongly collisional regime $\mu\ll 1$ reduces to

$$\langle \Delta x_p^2(t)\rangle_{b,\perp,\parallel}=\frac{2\chi_\perp}{\mu\nu}e^{\mu\nu t} . \quad (96)$$

We thus find a typical exponential departure of the particle trajectory from the initial field line $\sim\exp(\mu\nu t)$. The growth rate is related to the exponentiation length of the field line L_K, defined in Eq. (55): it introduces a positive temporal Lyapounov exponent T_K identified by the relation $\exp(\mu\nu t)\equiv\exp(2t/T_K)$. In the limit of a large $\lambda_\perp$ it is approximated by using Eqs. (57) and (72):

$$\nu T_K=\frac{2}{\mu}\approx\frac{\nu^2\lambda_\perp^4}{8\pi\beta^4\lambda_\parallel^2V_T^2} . \quad (97)$$

A very important feature appearing in Eq. (96) is that the trajectory-field line decorrelation can only occur if $\chi_\perp\neq 0$. This role of the perpendicular collisional diffusion was already stressed in Ref. [5]; a particularly clear qualitative discussion is found in Ref. [14]. We now note that the exponential separation cannot go on forever; the result (96), based on a linearized equation, is valid only when $\langle \Delta x_p^2(t)\rangle_{b,\parallel,\perp}\lesssim\lambda_\perp^2$. Beyond that limit we may expect a diffusion process of the particles, independently of the field line. The situation is very much reminiscent of the "clump" problem of electrostatic drift wave turbulence [34,39]. We intend to return later with a deeper study of the nonlinear problem. Meanwhile, a semiquantitative estimate can be obtained as follows. The decorrelation process described here determines an effective random walk with a length step $\lambda_\perp$. The diffusion coefficient can then be evaluated as $D^{(2)}=\lambda_\perp^2/2\tau_d$, where the time step τ_d, called the decorrelation time, is the time interval during which $\langle \Delta x_p^2(t)\rangle_{b,\parallel,\perp}$ grows from 0 to $\lambda_\perp^2$. The total diffusion coefficient from Eq. (81) is obtained by adding to $D^{(2)}$ the contribution $D^{(1)}$. As the latter represents a subdiffusive process, $D^{(1)}=0$. The total diffusion coefficient D is thus

$$D=D^{(2)}=\frac{2\chi_{\parallel}\lambda_{\perp}^2}{L_K^2\ln\left\{2\left[\frac{\lambda_{\perp}}{L_K}\right]^2\frac{\chi_{\parallel}}{\chi_{\perp}}\right\}} . \quad (98)$$

This is practically identical to the celebrated Rechester-Rosenbluth formula for the effective diffusion coefficient in the quasilinear limit ($\beta\lambda_{\parallel}/\lambda_{\perp}\ll 1$). Indeed, in the form given in Eq. (23) of Ref. [14], the latter coefficient is

$$D_{\mathrm{RR}}=\frac{\chi_{\parallel}D_m}{L_K\ln\left\{\left[\frac{\lambda_{\perp}}{L_K}\right]^2\frac{\chi_{\parallel}}{\chi_{\perp}}\right\}}=\frac{\chi_{\parallel}\lambda_{\perp}^2}{4L_K^2\ln\left\{\left[\frac{\lambda_{\perp}}{L_K}\right]^2\frac{\chi_{\parallel}}{\chi_{\perp}}\right\}} , \quad (99)$$

where we made use of Eq. (57) for L_K and Eq. (41) for $D_m \approx D_{\mathrm{QL}}$.

Summarizing our approximations, the domain of validity for the diffusion coefficient (98) can be determined. The parallel motion was considered to be diffusive: it defines a characteristic length in the z direction along which the decorrelation is achieved; this length is of the order $L_d^2 \approx \langle z_p^2(\tau_d)\rangle_{\parallel} \approx 2\chi_{\parallel}\tau_d$. L_d must be larger than the Kolmogorov length. When this condition is satisfied, the small initial cross-field displacement is significantly amplified. The characteristic lengths are ordered as follows in this strongly collisional regime:

$$L_{\mathrm{mfp}} \ll L_K \ll L_d, \quad \lambda_{\parallel} < L_K . \quad (100)$$

The second inequality takes into account the criterion (58) for the validity of the Markovian approximation. Moreover, Eq. (95) shows that the perpendicular collisional diffusivity $\chi_{\perp}$ contributes linearly to the particle-field line decorrelation. This implies an additional condition for ensuring that the decorrelation mechanism due to the chaotic field prevails over the collisional cross-field displacement. This condition can be expressed in terms of the "Kadomtsev-Pogutse characteristic length" L_{KP}

$$L_d < L_{\mathrm{KP}} \equiv \lambda_{\perp}\sqrt{\chi_{\parallel}/\chi_{\perp}} . \quad (101)$$

Thus the conditions (100) and (101) for which the diffusion coefficient (98) was evaluated define indeed the strongly collisional validity domain for the Rechester-Rosenbluth regime.

2. *The limit of weak collisionality*

We can also consider the weakly collisional limit $\mu \gg 1$, in which the particle-field line decorrelation is produced during the ballistic regime of the parallel motion: this implies $\tau_d < \nu^{-1}$. During the finite time interval $t<\tau_d$, we approximate $\varphi(\nu t)\approx \nu t$ and $\psi(\nu t)\approx \frac{1}{2}(\nu t)^2$ and obtain (asymptotically) from Eq. (94)

$$\langle \Delta x_p^2(t)\rangle_{b,\perp,\parallel}=2\chi_{\perp}\nu\int_0^t dt_1(t-t_1)e^{(1/2)\mu\nu^2 t_1^2} \sim \frac{2\chi_{\perp}}{\nu}\frac{1}{\mu^2(\nu t)^2}\exp\left[\frac{\mu}{2}(\nu t)^2\right] . \quad (102)$$

Thus the mean square particle-field line separation grows like the exponential of t^2. (A similar behavior was found in Ref. [34] for electrostatic drift wave turbulence.) The decorrelation time is determined from the equation $\langle \Delta x_p^2(\tau_d)\rangle_{b\perp\parallel}=\lambda_{\perp}^2$, which is of transcendental type. We found that, for a rather wide range of parameters, the solution of this equation is approximated by

$$\nu\tau_d=\left[\frac{2}{\mu}\right]^{1/2}\ln^{1/2}\left[\frac{4L_{\mathrm{KP}}^2}{L_K^2}\right] , \quad (103)$$

which yields the following estimate for the random walk diffusion coefficient in the weakly collisional regime:

$$D_{\mathrm{weak}}^{(2)}=\frac{2D_m V_T}{\ln^{1/2}\left[\frac{4L_{\mathrm{KP}}^2}{L_K^2}\right]}=\frac{2D_m V_T}{\ln^{1/2}\left[2\mu\frac{\lambda_{\perp}^2}{\rho_L^2}\right]} . \quad (104)$$

The domain of validity of this result is

$$\lambda_{\parallel} < L_K < L_d \ll L_{\mathrm{mfp}}, \quad L_d < L_{\mathrm{KP}} , \quad (105)$$

where the characteristic length for particle decorrelation along the magnetic field L_d is of the order $L_d \approx V_T\tau_d$.

The total diffusion coefficient is obtained by adding the two terms in Eq. (81). We now recall that the contribution $D^{(1)}$, Eq. (91), of the first term is subdiffusive in all collisional regimes, except in the strictly collisionless case $\nu=0$ ($\mu=\infty$), Eq. (91). We thus obtain the following form for the weakly collisional diffusion coefficient:

$$D_{\mathrm{weak}}=\begin{cases}\dfrac{2D_m V_T}{\ln^{1/2}\left[2\mu\dfrac{\lambda_{\perp}^2}{\rho_L^2}\right]}, & \nu\neq 0\\[2ex] \dfrac{1}{\sqrt{\pi}}D_m V_T, & \nu=0 .\end{cases} \quad (106)$$

The diffusion coefficient (106) was not previously derived. It describes, like the Rechester-Rosenbluth coefficient (98), the amplification of a small initial cross-field displacement in the chaotic field, but applies to weakly collisional plasmas. The coefficient of the first line ($D^{(2)}$) is a function of the collision frequency. In the limit $\nu\to 0$, i.e., $\mu\to\infty$, this term decreases slowly to zero. When ν is exactly zero, the particles can no longer decorrelate from the field lines (in the present model) and $D_{\nu=0}^{(2)}=0$. But in the same case, the term $D^{(1)}$ becomes suddenly diffusive: the particles that stick to the field lines are dragged in the perpendicular direction by the diffusion of the lines. As a result, there appears a finite, well known "collisionless diffusion coefficient" (91) [40]. [It may be noted that in Refs. [11,14,21,40] the coefficient is 2 instead of $\pi^{-1/2}$. The former coefficient is obtained from the qualitative argument $\langle \delta x^2(t)\rangle$

$=2D_m z(t) \approx 2D_m V_T t$. Instead of injecting the approximate value $V_T t$ into an asymptotic approximation, we calculate here in a single stroke the exact asymptotic limit of Eq. (87), for a given spectrum (11): this is the origin of the slightly different numerical coefficient.]

The collisionless limit appears to be rather singular. It was already mentioned in earlier work (see, in particular, the critique in Ref. [11] and the discussion in Sec. VIII and Fig. 1 of Ref. [27]) that the collisionless "limit" is not a well-defined concept. We believe that our present result clarifies this question. Equation (106) exhibits the quite distinct physical role of the two terms contributing to the diffusion coefficient. Note that the two running diffusion coefficients Eq. (80) are finite quantities for all times and for all collision frequencies. But for a strictly zero collisionality, the coefficient $D^{(1)}(t)$ presents a sudden diffusive behavior and produces, by itself, a nonzero diffusion coefficient. The coefficient $D_0^{(1)}$ cannot be attained by a continuous limiting procedure. On the other hand, the strictly collisionless case is physically fictitious. Therefore the "collisionless diffusion coefficient" $D_0^{(1)}$ can never be reached and cannot represent an approximation acceptable for weakly collisional systems, as appears clearly from Fig. 1 of Ref. [27].

It should be noted, on the other hand, that in the regime of very weak collisionality, the inefficient decorrelating action of the collisions will be superseded by other decorrelating mechanisms that are neglected in the present model, such as the perpendicular drift motions caused by magnetic field gradients and curvature. When these factors are considered, the transition to the collisionless limit is no longer so sudden. These effects will be studied in a forthcoming work.

Laval [21] obtained an expression for the diffusion coefficient by a very different method, using a stochastic dynamics modeled by a discrete map, related to the sawtooth map. His results, which depend on his specific model, can hardly be compared to ours. In this particular model he obtains, in the strongly collisional limit, a result (his Eq. 42) very similar (but not identical) to the Rechester-Rosenbluth formula. His general result reduces, in the weakly collisionless limit, to $2D_m V_T$. But we cannot see in his formula the distinct behavior of the two terms of Eq. (79), which is so peculiar in our result. In view of the discussion presented above, we believe that Laval's "continuous limit" towards a collisionless diffusion coefficient is open to question.

VII. CONCLUSIONS

Our first purpose in the present paper was the precise definition of the main concepts entering the statistical representation of the fluctuating magnetic field. Given that the magnetic field is modeled as a Gaussian process, all its moments are determined by the two-point correlation functions of the magnetic field. One can distinguish two types of such correlation functions, which pose very different problems.

The Eulerian correlations are simple objects, entirely determined by the kinematics, or geometry of the situation. One must, however, be careful in taking account of all constraints, in particular the zero-divergence constraint [19]. We have given here detailed formulas for these objects. It turns out that the explicit expressions given by some authors for these quantities are incorrect because they are inconsistent with the geometrical constraints.

The Lagrangian correlations, by contrast, are very complex objects, the determination of which requires the solution of the equations of motion. One must therefore accept reasonable approximations such as the Corrsin approximation used here. It is shown that within this framework the Lagrangian correlations can be reduced to a single scalar function, which is determined by an integral equation. The latter can easily be solved numerically. This treatment of the Lagrangian correlations appears to be different.

The results obtained for the Lagrangian correlation immediately lead to the derivation of a second-order nonlinear differential equation for the mean square deviation of a (geometrical) point running along a magnetic field line. The study of diffusion problems through the solution of a differential equation is a rather new methodology (the only other similar but incomplete treatment, to the best of our knowledge, appears in an unpublished report of Rax and White [41]). It leads, in particular, to an analytic expression of the diffusion coefficient, both in the quasilinear limit and in the opposite, percolation limit. In between, the numerical solution of the equation presents no problems. It exhibits a transition from a regime proportional to α^2 to one proportional to α: this is very reminiscent of the transition from quasilinear to Bohm-like regimes in electrostatic drift wave turbulence. It is shown, however, that the large-α limit (i.e., the Bohm-like form) is actually illusory.

The two problems treated in Secs. V and VI have a common feature. In both cases there appears an exponential, and thus chaotic, separation of two initially very close curves. The difference is in the nature of these curves.

In Sec. V we treated the purely geometrical problem of separation of two magnetic field lines in a spatially fluctuating magnetic field. The latter is described by a Gaussian process with spatial correlation scales $\lambda_\parallel$ ($\lambda_\perp$) in the parallel (perpendicular) direction, respectively. We stressed the different picture obtained in this process, according to whether one looks at the figure obtained in a single realization or in the average over an ensemble of realizations. In the former case an initial elliptic flux tube tends toward a complex shape due to stretching and narrowing in different directions as the points advance along the lines [14]. In the average picture, there is, on the contrary, a tendency toward gyrotropization; the ellipse tends toward a circle and any initial anisotropy disappears in the long range.

Another important aspect exhibited here is the importance of correctly taking into account the constraint of zero divergence of the magnetic field in assessing the form of the magnetic field correlations. Surprisingly, this aspect was not fully considered in earlier work [14]. The consequences are not only minor changes in the numerical factors in the expression of the exponentiation lengths: the signs of the latter are changed; this leads to a completely different picture of the evolution.

The problem treated in Sec. VI is a truly dynamical one, in which the evolution in time is considered. We study here the exponential separation of a particle from the magnetic field line to which it was initially tied. This problem led us to the evaluation of an anomalous perpendicular diffusion coefficient. For a strongly collisional plasma, this coefficient reproduces the Rechester-Rosenbluth diffusion coefficient. We were also able to treat the weakly collisional limit, thus obtaining an alternative form of the anomalous diffusion coefficient, valid in this domain. The detailed discussion of the strictly collisionless case contributes toward the clarification of this strange and controversial problem.

It may be stressed here that the Rechester-Rosenbluth coefficient was never before obtained by a complete mathematical solution of the problem, but rather was based on semi-intuitive arguments. The latter (especially in Isichenko's version [14]) were rather convincing and picturesque: the particle was described as traveling a long way along the field lines of an initial flux tube until it reaches a region where the exponential narrowing of the tube (along the stable manifold) is so strong that the small collisional diffusion is sufficient for making it leave the tube.

Our own treatment is, from the start, more quantitative: it involves the solution of the basic equation of evolution of the problem. The latter is, however, approximated by its linearized version, which is valid for times shorter than the decorrelation time τ_d. It is during this initial phase that we find the exponential separation. Up to this point our treatment is purely deductive. The final result, rather than being deduced mathematically, results from the assumption of a final diffusive phase. The full nonlinear treatment of the equation is short-circuited by introducing the picture of a random walk of step size $\lambda_\perp$. Calculating the diffusion coefficient associated with this random walk, we attain the Rechester-Rosenbluth result (in the strongly collisional domain).

In conclusion, it is believed that our treatment of the anomalous transport across the magnetic field, due to magnetic fluctuations, represents a relevant, though incomplete, step toward a theory based on first principles. We continue the work on the next (nonlinear) step and expect to return to this problem in forthcoming papers.

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APPENDIX: THE TENSOR OF MEAN SQUARE DISPLACEMENTS

We define the following tensor of mean square displacements generalizing Eq. (22):

$$\Gamma_{mn}(\zeta)\equiv\langle\delta x_m(\zeta)\delta x_n(\zeta)\rangle = \int_0^\zeta d\zeta_1\int_0^\zeta d\zeta_2\mathcal{L}_{mn}(\zeta_1,\zeta_2) = 2\int_0^\zeta d\zeta'(\zeta-\zeta')\mathcal{L}_{mn}(\zeta')\ , \qquad (A1)$$

where we used the stationarity and the symmetry of the Lagrangian correlation tensor $\mathcal{L}_{mn}(\zeta)$, but assumed nothing about its specific form. Double differentiation with respect to ζ yields

$$\frac{d^2\Gamma_{mn}(\zeta)}{d\zeta^2}=2\mathcal{L}_{mn}(\zeta) =2\int d\boldsymbol{k}_\perp dk_\parallel \mathcal{B}_{mn}(\boldsymbol{k}_\perp,k_\parallel)e^{ik_\parallel\zeta} \times\exp[-\tfrac{1}{2}k_r k_s\Gamma_{rs}(\zeta)]\ . \qquad (A2)$$

We thus obtain a set of three coupled equations for the components of the MSD tensor. The wave-vector integrations are easily performed. Using the dimensionless quantities defined in Eq. (32) and the dimensionless tensor $g_{mn}(\tau)=\lambda_\perp^{-2}\Gamma_{mn}(\lambda_\parallel\tau)$, we obtain

$$\frac{d^2}{d\tau^2}g_{mn}(\tau)=\frac{2\alpha^2 e^{-\tau^2/2}}{\{[1+g_{xx}(\tau)][1+g_{yy}(\tau)]-g_{xy}^2(\tau)\}^{3/2}} \times[\delta_{mn}+g_{mn}(\tau)]\ , \qquad (A3)$$

with the initial conditions derived from Eq. (A1):

$$g_{mn}(0)=\left.\frac{dg_{mn}}{d\tau}\right|_{\tau=0}=0\ . \qquad (A4)$$

We note that the equations for g_{xy} and g_{yx} are homogeneous; hence the initial condition (A4) implies

$$g_{xy}(\tau)=g_{yx}(\tau)=0,\quad \tau>0\ . \qquad (A5)$$

Next we note that the combination $[g_{xx}(\tau)-g_{yy}(\tau)]$ also obeys a homogeneous equation; hence

$$g_{xx}(\tau)=g_{yy}(\tau)\equiv g(\tau),\quad \tau>0\ . \qquad (A6)$$

We thus proved that the MSD tensor is proportional to the unit tensor

$$g_{mn}(\tau)=g(\tau)\delta_{mn}\ . \qquad (A7)$$

It is easily seen from Eq. (A1) that Eq. (A7) induces the same property for the Lagrangian correlation $\mathcal{L}_{mn}(\zeta)=\mathcal{L}(\zeta)\delta_{mn}$ and thus we obtained here a proof of Eq. (28).

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